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Research Article

L^p Approximation by Multivariate Baskakov-Durrmeyer Operator

Feilong Cao and Yongfeng An

Department of Mathematics, China Jiliang University, Hangzhou 310018, Zhejiang Province, China

Correspondence should be addressed to Feilong Cao, feilongcao@gmail.com

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The main aim of this paper is to introduce and study multivariate Baskakov-Durrmeyer operator, which is nontensor product generalization of the one variable. As a main result, the strong direct inequality of L^p approximation by the operator is established by using a decomposition technique.

1. Introduction

Let $P_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$, $x \in [0, \infty)$, $n \in \mathbb{N}$. The Baskakov operator defined by

$$B_{n,1}(f, x) = \sum_{k=0}^{\infty} P_{n,k}(x) f\left(\frac{k}{n}\right) \quad (1.1)$$

was introduced by Baskakov [1] and can be used to approximate a function f defined on $[0, \infty)$. It is the prototype of the Baskakov-Kantorovich operator (see [2]) and the Baskakov-Durrmeyer operator defined by (see [3, 4])

$$M_{n,1}(f, x) = \sum_{k=0}^{\infty} P_{n,k}(x) (n-1) \int_0^{\infty} P_{n,k}(t) f(t) dt, \quad x \in [0, \infty), \quad (1.2)$$

where $f \in L^p[0, \infty)$ ($1 \leq p < \infty$).

By now, the approximation behavior of the Baskakov-Durrmeyer operator is well understood. It is characterized by the second-order Ditzian-Totik modulus (see [3])

$$\omega_{\varphi}^2(f, t)_p = \sup_{0 < h \leq t} \|f(\cdot + 2h\varphi(\cdot)) - 2f(\cdot + h\varphi(\cdot)) + f(\cdot)\|_p, \quad \varphi(x) = \sqrt{x(1+x)}. \quad (1.3)$$

More precisely, for any function defined on $L^p[0, \infty)$ ($1 \leq p < \infty$), there is a constant such that

$$\|M_{n,1}(f) - f\|_p \leq \text{const.} \left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_p + \frac{1}{n} \|f\|_p \right), \quad (1.4)$$

$$\omega_\varphi^2(f, t)_p = O(t^{2\alpha}) \iff \|M_{n,1}(f) - f\|_p = O(n^{-\alpha}), \quad (1.5)$$

where $0 < \alpha < 1$.

Let $T \subset \mathbb{R}^d$ ($d \in \mathbb{N}$), which is defined by

$$T := T_d := \{\mathbf{x} := (x_1, x_2, \dots, x_d) : 0 \leq x_i < \infty, 1 \leq i \leq d\}. \quad (1.6)$$

Here and in the following, we will use the standard notations

$$\begin{aligned} \mathbf{x} &:= (x_1, x_2, \dots, x_d), & \mathbf{k} &:= (k_1, k_2, \dots, k_d) \in \mathbb{N}_0^d, \\ \mathbf{x}^{\mathbf{k}} &:= x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}, & \mathbf{k}! &:= k_1! k_2! \cdots k_d!, & |\mathbf{x}| &:= \sum_{i=1}^d x_i, & |\mathbf{k}| &:= \sum_{i=1}^d k_i, \\ \binom{n}{\mathbf{k}} &:= \frac{n!}{\mathbf{k}!(n - |\mathbf{k}|)!}, & \sum_{\mathbf{k}=0}^\infty &:= \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \cdots \sum_{k_d=0}^\infty. \end{aligned} \quad (1.7)$$

By means of the notations, for a function f defined on T the multivariate Baskakov operator is defined as (see [5])

$$B_{n,d}(f, \mathbf{x}) := \sum_{\mathbf{k}=0}^\infty f\left(\frac{\mathbf{k}}{n}\right) P_{n,\mathbf{k}}(\mathbf{x}), \quad (1.8)$$

where

$$P_{n,\mathbf{k}}(\mathbf{x}) = \binom{n + |\mathbf{k}| - 1}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 + |\mathbf{x}|)^{-n - |\mathbf{k}|}. \quad (1.9)$$

Naturally, we can modify the multivariate Baskakov operator as multivariate Baskakov-Durrmeyer operator

$$M_{n,d}f := M_{n,d}(f, \mathbf{x}) := \sum_{\mathbf{k}=0}^\infty P_{n,\mathbf{k}}(\mathbf{x}) \phi_{n,\mathbf{k},d}(f), \quad f \in L^p(T), \quad (1.10)$$

where

$$\phi_{n,\mathbf{k},d}(f) := \frac{\int_T P_{n,\mathbf{k}}(\mathbf{u}) f(\mathbf{u}) d\mathbf{u}}{\int_T P_{n,\mathbf{k}}(\mathbf{u}) d\mathbf{u}} = (n-1)(n-2) \cdots (n-d) \int_T P_{n,\mathbf{k}}(\mathbf{u}) f(\mathbf{u}) d\mathbf{u}. \quad (1.11)$$

It is a multivariate generalization of the univariate Baskakov-Durrmeyer operators given in (1.2) and can be considered as a tool to approximate the function in $L^p(T)$.

2. Main Result

We will show a direct inequality of L^p approximation by the Baskakov-Durrmeyer operator given in (1.10). By means of K -functional and modulus of smoothness defined in [5], we will extend (1.4) to the case of higher dimension by using a decomposition technique.

For $\mathbf{x} \in T$, we define the weight functions

$$\varphi_i(\mathbf{x}) = \sqrt{x_i(1 + |\mathbf{x}|)}, \quad 1 \leq i \leq d. \quad (2.1)$$

Let

$$D_i^r = \frac{\partial^r}{\partial x_i^r}, \quad r \in \mathbb{N}, \quad D^{\mathbf{k}} = D_1^{k_1} D_2^{k_2} \cdots D_d^{k_d}, \quad \mathbf{k} \in \mathbb{N}_0^d \quad (2.2)$$

denote the differential operators. For $1 \leq p < \infty$, we define the weighted Sobolev space as follows:

$$W_\varphi^{r,p}(T) = \left\{ f \in L^p(T) : D^{\mathbf{k}} f \in L_{\text{loc}}(T), \varphi_i^r D_i^r f \in L^p(T) \right\}, \quad (2.3)$$

where $|\mathbf{k}| \leq r$, $\mathbf{k} \in \mathbb{N}_0^d$, and T denotes the interior of T . The Peetre K -functional on $L^p(T)$ ($1 \leq p < \infty$), are defined by

$$K_\varphi^r(f, t^r)_p = \inf \left\{ \|f - g\|_p + t^r \sum_{i=1}^d \|\varphi_i^r D_i^r g\|_p \right\}, \quad t > 0, \quad (2.4)$$

where the infimum is taken over all $g \in W_\varphi^{r,p}(T)$.

For any vector \mathbf{e} in \mathbb{R}^d , we write the r th forward difference of a function f in the direction of \mathbf{e} as

$$\Delta_{h\mathbf{e}}^r f(\mathbf{x}) = \begin{cases} \sum_{i=0}^r \binom{r}{i} (-1)^i f(\mathbf{x} + i h \mathbf{e}), & \mathbf{x}, \mathbf{x} + r h \mathbf{e} \in T, \\ 0, & \text{otherwise.} \end{cases} \quad (2.5)$$

We then can define the modulus of smoothness of $f \in L^p(T)$ ($1 \leq p < \infty$), as

$$\omega_\varphi^r(f, t)_p = \sup_{0 < h \leq t} \sum_{i=1}^d \|\Delta_h^r \varphi_i \mathbf{e}_i f\|_p, \quad (2.6)$$

where \mathbf{e}_i denotes the unit vector in \mathbb{R}^d , that is, its i th component is 1 and the others are 0.

In [5], the following result has been proved.

Lemma 2.1. *There exists a positive constant, dependent only on p and r , such that for any $f \in L^p(T)$, $1 \leq p < \infty$*

$$\frac{1}{\text{const.}} \omega_\varphi^r(f, t)_p \leq K_\varphi^r(f, t^r)_p \leq \text{const.} \omega_\varphi^r(f, t)_p. \quad (2.7)$$

Now we state the main result of this paper.

Theorem 2.2. *If $f \in L^p(T)$, $1 \leq p < \infty$, then there is a positive constant independent of n and f such that*

$$\|M_{n,d}f - f\|_p \leq \text{const.} \left(\omega_\varphi^2 \left(f, \frac{1}{\sqrt{n}} \right)_p + \frac{1}{n} \|f\|_p \right). \quad (2.8)$$

Proof. Our proof is based on an induction argument for the dimension d . We will also use a decomposition method of the operator $M_{n,d}f$. We report the detailed proof only for two dimensions. The higher dimensional cases are similar.

Our proof depends on Lemma 2.1 and the following estimates:

$$\|M_{n,2}f - f\|_p \leq \text{const.} \begin{cases} \|f\|_{p'}, & f \in L^p(T), \\ \frac{1}{n} \left(\sum_{i=1}^2 \|\varphi_i^2 D_i^2 f\|_p + \|f\|_p \right), & f \in W_\varphi^{2,p}(T). \end{cases} \quad (2.9)$$

The first estimate is evident as the $M_{n,d}f$ are positive and linear contractions on $L^p(T)$ ($1 \leq p < \infty$). We can demonstrate the second estimate by reducing it to the one dimensional inequality

$$\|M_{n,1}f - f\|_p \leq \frac{\text{const.}}{n} \left(\|\varphi^2 f''\|_p + \|f\|_p \right), \quad (2.10)$$

which has been proved in [3]

Now we give the following decomposition formula:

$$\begin{aligned} M_{n,2}(f, \mathbf{x}) &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P_{n,k_1}(x_1) P_{n+k_1,k_2} \left(\frac{x_2}{1+x_1} \right) (n-1)(n-2) \\ &\quad \times \int_0^\infty \int_0^\infty P_{n,k_1}(u_1) P_{n+k_1,k_2} \left(\frac{u_2}{1+u_1} \right) f(u_1, u_2) du_1 du_2 \\ &= \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1) (n-2) \int_0^\infty P_{n-1,k_1}(u_1) \sum_{k_2=0}^{\infty} P_{n+k_1,k_2} \left(\frac{x_2}{1+x_1} \right) \\ &\quad \times (n+k_1-1) \int_0^\infty P_{n+k_1,k_2}(t) f(u_1, (1+u_1)t) dt du_1 \\ &= \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1) (n-2) \int_0^\infty P_{n-1,k_1}(u_1) M_{n+k_1,1}(g_{u_1}, z) du_1, \end{aligned} \quad (2.11)$$

where

$$g_{u_1}(t) = f(u_1, (1 + u_1)t), \quad 0 \leq t < \infty, \quad z = \frac{x_2}{1 + x_1}, \quad (2.12)$$

which can be checked directly and will play an important role in the following proof.

From the decomposition formula, it follows that

$$\begin{aligned} M_{n,2}(f, \mathbf{x}) - f(\mathbf{x}) &= \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1)(n-2) \\ &\quad \times \left\{ \int_0^{\infty} P_{n-1,k_1}(u_1)(M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)) du_1 \right\} + M_{n,1}^*(h(\cdot), x_1) - h(x_1) \\ &:= J + L, \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} h(u_1) &:= h(u_1, \mathbf{x}) := f\left(u_1, (1 + u_1)\frac{x_2}{1 + x_1}\right), \quad 0 \leq u_1 < \infty, \\ M_{n,1}^*(g, y) &= \sum_{l=0}^{\infty} P_{n,l}(y)(n-2) \int_0^{\infty} P_{n-1,l}(t)g(t)dt. \end{aligned} \quad (2.14)$$

Then by the Jensen's inequality, we have

$$\begin{aligned} \|J\|_p^p &\leq \int_T \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1) \left| (n-2) \int_0^{\infty} P_{n-1,k_1}(u_1)(M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)) du_1 \right|^p dx \\ &\leq \int_T \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1)(n-2) \int_0^{\infty} P_{n-1,k_1}(u_1) |M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)|^p du_1 dx \\ &= \int_0^{\infty} \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1)(1 + x_1) dx_1 (n-2) \int \int_0^{\infty} P_{n-1,k_1}(u_1) \\ &\quad \times |M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)|^p dz du_1 \\ &\leq \sum_{k_1=0}^{\infty} \frac{n + k_1 - 1}{n - 1} \int_0^{\infty} P_{n-1,k_1}(u_1) \int_0^{\infty} |M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)|^p dz du_1 \\ &\leq \text{const.} \sum_{k_1=0}^{\infty} \frac{n + k_1 - 1}{n - 1} \int_0^{\infty} P_{n-1,k_1}(u_1) \left(\frac{1}{n + k_1}\right)^p \left(\|\varphi^2 g_{u_1}''\|_p^p + \|g_{u_1}\|_p^p\right) du_1. \end{aligned} \quad (2.15)$$

However, by definition, one also has

$$\varphi^2(t)g_{u_1}''(t) = t(1+t)(1+u_1)^2 D_2^2 f(u_1, (1+u_1)t) = (\varphi_2^2 D_2^2 f)(u_1, (1+u_1)t). \quad (2.16)$$

Therefore,

$$\begin{aligned}
\|J\|_p^p &\leq \text{const.} \sum_{k_1=0}^{\infty} \frac{n+k_1-1}{(n-1)(n+k_1)^p} \int_0^{\infty} P_{n-1,k_1}(u_1) \\
&\quad \times \left(\left| (\varphi_2^2 D_2^2 f)(u_1, (1+u_1)t) \right|^p + |f(u_1, (1+u_1)t)|^p \right) dt du_1 \\
&= \text{const.} \sum_{k_1=0}^{\infty} \frac{n+k_1-1}{(n-1)(n+k_1)^p} \int_0^{\infty} \frac{1}{1+u_1} P_{n-1,k_1}(u_1) \\
&\quad \times \int_0^{\infty} \left(\left| (\varphi_2^2(u_1, u_2) D_2^2 f)(u_1, u_2) \right|^p + |f(u_1, u_2)|^p \right) du_1 du_2 \\
&\leq \frac{\text{const.}}{n^p} \sum_{k_1=0}^{\infty} \int_0^{\infty} P_{n,k_1}(u_1) \int_0^{\infty} \left(\left| (\varphi_2^2(u_1, u_2) D_2^2 f)(u_1, u_2) \right|^p + |f(u_1, u_2)|^p \right) du_1 du_2 \\
&= \frac{\text{const.}}{n^p} \left(\left\| \varphi_2^2 D_2^2 f \right\|_p^p + \|f\|_p^p \right).
\end{aligned} \tag{2.17}$$

To estimate the second term L , we use a similar method as to estimate (2.10) (see [3]) and can get

$$\|L\|_p \leq \frac{\text{const.}}{n} \left(\left\| \varphi^2 h'' \right\|_p + \|h\|_p \right). \tag{2.18}$$

Denoting $\varphi_{12}(\mathbf{x}) = \varphi_{21}(\mathbf{x}) := \sqrt{x_1 x_2}$, $D_{12}^2 := \partial^2 / (\partial x_1 \partial x_2)$, and $D_{21}^2 := \partial^2 / (\partial x_2 \partial x_1)$, we have

$$\begin{aligned}
&\left| \varphi^2(s) h''(s) \right| \\
&= \left| s(1+s) \left(D_1^2 f + \frac{x_2}{1+x_1} D_{12}^2 f + \frac{x_2}{1+x_1} D_{21}^2 f + \frac{x_2^2}{(1+x_1)^2} D_{22}^2 f \right) \times \left(s, (1+s) \frac{x_2}{1+x_1} \right) \right| \\
&= \left| \left(\frac{1+x_1}{1+x_1+x_2} \varphi_1^2 D_1^2 f + \varphi_{12}^2 D_{12}^2 f + \varphi_{21}^2 D_{21}^2 f + \frac{s}{1+s} \frac{x_2}{1+x_1+x_2} \varphi_2^2 D_2^2 f \right) \left(s, (1+s) \frac{x_2}{1+x_1} \right) \right|.
\end{aligned} \tag{2.19}$$

Recalling that $\varphi_{12}(\mathbf{x})$ is no bigger than $\varphi_1(\mathbf{x})$ or $\varphi_2(\mathbf{x})$, and the fact

$$\left| D_{12}^2 f(\mathbf{x}) \right| \leq \sup \left(\left| D_1^2 f(\mathbf{x}) \right|, \left| D_2^2 f(\mathbf{x}) \right| \right) \tag{2.20}$$

proved in [6] (see [6, Lemma 2.1]), we obtain

$$\left\| \varphi^2 h'' \right\|_p \leq \text{const.} \sum_{i=1}^2 \left\| \varphi_i^2 D_i^2 f \right\|_p, \tag{2.21}$$

and hence

$$\|L\|_p \leq \frac{\text{const.}}{n} \left(\sum_{i=1}^2 \left\| \varphi_i^2 D_i^2 f \right\|_p + \|f\|_p \right). \quad (2.22)$$

The second inequality of (2.9) has thus been established, and the proof of Theorem 2.2 is finished. \square

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References

- [1] V. A. Baskakov, "An instance of a sequence of linear positive operators in the space of continuous functions," *Doklady Akademii Nauk SSSR*, vol. 113, pp. 249–251, 1957.
- [2] Z. Ditzian and V. Totik, *Moduli of Smoothness*, vol. 9 of *Springer Series in Computational Mathematics*, Springer, New York, NY, USA, 1987.
- [3] M. Heilmann, "Direct and converse results for operators of Baskakov-Durrmeyer type," *Approximation Theory and its Applications*, vol. 5, no. 1, pp. 105–127, 1989.
- [4] A. Sahai and G. Prasad, "On simultaneous approximation by modified Lupas operators," *Journal of Approximation Theory*, vol. 45, no. 2, pp. 122–128, 1985.
- [5] F. Cao, C. Ding, and Z. Xu, "On multivariate Baskakov operator," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 1, pp. 274–291, 2005.
- [6] W. Chen and Z. Ditzian, "Mixed and directional derivatives," *Proceedings of the American Mathematical Society*, vol. 108, no. 1, pp. 177–185, 1990.